

## Effect of initial conditions on Glauber dynamics in complex networks

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(Received 7 July 2006; revised manuscript received 9 November 2006; published 6 April 2007)

The effect of initial spin configurations on zero-temperature Glauber spin dynamics in complex networks is investigated. In a system in which the initial spins are defined by centrality measures at the vertices of a network, a variety of nontrivial diffusive behaviors arise, particularly in relation to functional relationships between the initial and final fractions of positive spins, some of which exhibit a critical point. Notably, the majority spin in the initial state is not always dominant in the final state and the phenomena that occur as a result of the dynamics differ according to the initial condition, even for the same network. It is thus concluded that the initial condition of a complex network exerts an influence on spin dynamics that is equally as strong as that exerted by the network structure.

DOI: [10.1103/PhysRevE.75.046105](https://doi.org/10.1103/PhysRevE.75.046105)

PACS number(s): 89.75.Fb, 05.50.+q, 64.60.Cn, 89.75.Hc

The study of complex networks has generated broad interest, particularly in the field of nonlinear physics [1,2]. Dynamic processes and the dependence of such processes on networks are an important topic in the field of complex networks, and many situations that result in nontrivial global phenomena have been discovered, despite the relative simplicity of local interactions. The Glauber dynamics of the Ising model, involving a set of spinlike binary variables and corresponding local interaction, is some of the simplest dynamics that occur in such networks. The Ising model with Glauber dynamics has been investigated as a model of the ordering process in spin glasses and other systems [3]. The interaction patterns of Glauber dynamics on regular  $d$ -dimensional lattices and complex network structures have been considered in a number of studies, and it has been shown that such structures promote the development of characteristic ordering from a completely disordered state [4–8].

Yet beyond the effect of structural complexity itself, complex networks also exhibit complexity related to the dynamics peculiar to complex structures. Previous studies of Glauber dynamics have predominantly focused on the ordering dynamics that emerges from the completely disordered state—that is, the state in which the initial spins are randomly distributed with an initial fraction  $r$  of positive spins. However, the vertices in complex networks are not interconnected uniformly; instead, they form a heterogeneous distribution of their degrees or locations. As a result of this heterogeneity, vertices in networks must be characterized topologically, such as by the measure of *centrality* [9], which represents a set of parameters for individual vertices that are determined according to network topologies. If vertices are discriminated by centrality measures, an initial configuration can be considered that is not randomly distributed, but rather topologically biased. In such a case, it becomes possible to introduce some form of ordering into an initially disordered state. Given initial conditions, Glauber dynamics can also be considered as controlling the diffusion of spins from arbitrarily selected vertices in the initial condition. Such a semi-

ordered initial condition, in addition to the effect of the network structure, is expected to have a significant effect on the resulting dynamics. An attempt to demonstrate this effect through exploration of a simple two-state diffusion process has been reported previously [10]. However, while the numerical results can be categorized into several classes, they are considered merely preliminary, since the model is unfamiliar and numerical experiments alone are insufficient to confirm such categorization.

In the present study, the dependence of zero-temperature Glauber dynamics, one of the simplest forms of Glauber dynamics, on the initial spin configuration is investigated by Monte Carlo simulations of several types of complex networks. The dynamics is taken from the arbitrary, semiordered initial spin configuration determined by the characteristics of the network structure.

Zero-temperature Glauber dynamics is considered here for the case of a complex network with spin variables  $\sigma = \pm 1$  located at the vertices of the network. The network is denoted by the adjacency matrix  $A_{ij}$ , which is defined such that  $A_{ij}=1$  if the vertices  $i$  and  $j$  are connected and  $A_{ij}=0$  otherwise. In the present case of an undirected network,  $A_{ij}$  is symmetric. The local field  $h_i(\tau)$  acting on vertex  $i$  during time step  $\tau$  due to the spins of the neighboring vertices of the vertex  $i$  is given by

$$h_i(\tau) = \sum_j^N A_{ij} \sigma_j(\tau) \quad (i = 1, \dots, N). \quad (1)$$

In the preceding study, a similar dynamics was investigated as a two-state diffusion process [10] by considering the following model:

$$\sigma_i(n+1) = \begin{cases} \text{sgn}\{h_i(n)\} & \text{if } h_i(n) \neq 0, \\ \sigma_i(n) & \text{if } h_i(n) = 0. \end{cases} \quad (2)$$

In this model, the spins of vertices are updated synchronously at each time step  $n$ ; that is, the spins at all vertices are updated simultaneously as  $n$  progresses. In the present study, a general Monte Carlo method is employed for numerical simulations and the spins of individual vertices are updated

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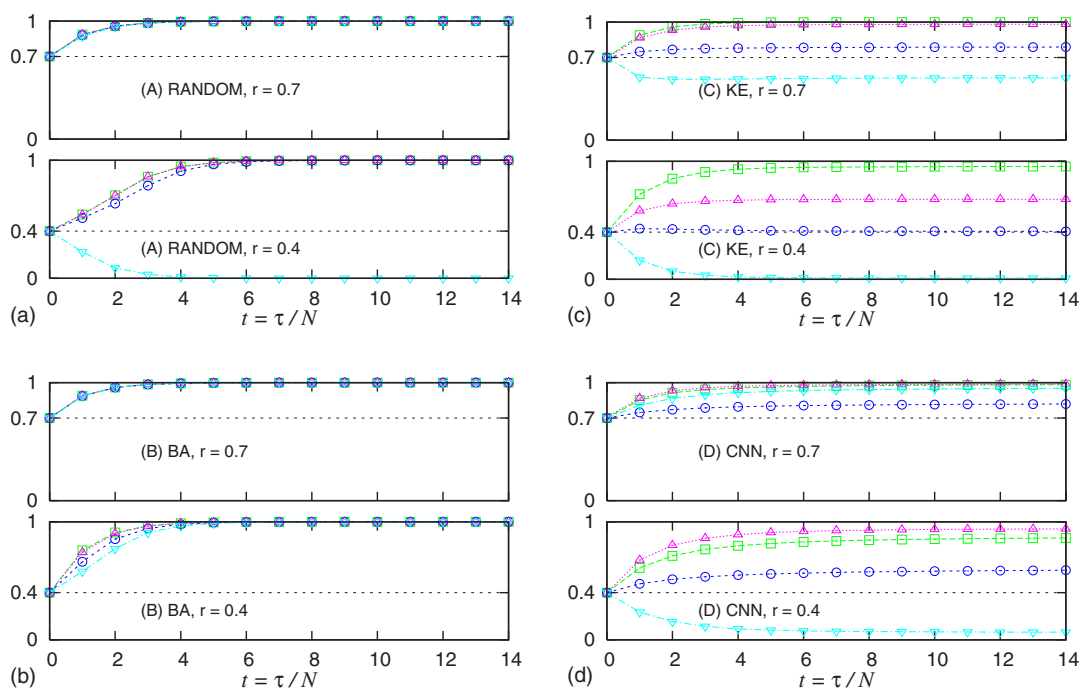


FIG. 1. (Color online) Temporal evolution of  $R_+(t)$  in (a) a random network (b), a BA network (c), a KE network, and (d) a CNN network (d). Initial fractions  $r=0.4$  and  $0.7$ . The different symbols denote the ordering of the initial state, according to degree centrality ( $\square$ ), closeness centrality ( $\circ$ ), betweenness centrality ( $\triangle$ ), and clustering coefficient ( $\nabla$ ).  $N=18000$  and  $\langle k \rangle=10$ .

asynchronously. The rule for updating the spin of a randomly selected vertex is as follows:

$$\sigma_i(\tau+1) = \begin{cases} \text{sgn}\{h_i(\tau)\} & \text{if } h_i(\tau) \neq 0, \\ \pm 1 (\text{probability } 1/2) & \text{if } h_i(\tau) = 0. \end{cases} \quad (3)$$

The progressive behavior of the spins is investigated using a normalized time step of  $t = \tau/N$ .

Spins on  $d$ -dimensional lattices have been investigated by a number of researchers, and it has been shown that despite the simplicity of zero-temperature Glauber dynamics, non-trivial phenomena arise under such regimes, even for regular lattices [11–13]. The Glauber dynamics of other types of complex networks, such as the Watts-Strogatz (WS) network [14], has also been examined. Boyer and Miramontes [6], by analyzing the ordering dynamics of the Watts-Strogatz network, revealed a nonequilibrium ordering process induced by “shortcuts” in the WS network, with occasional incomplete ordering. The random graph network of Erdős and Rényi [15] and scale-free networks (i.e., Barabási and Albert [16]) have also been investigated [7], for which the Glauber dynamics does not lead to a fully ordered state. In such systems, the dynamics may become trapped in a set of partially ordered (meta)stable states, even when the system is finite. Zhou and Lipowsky [17] described a mean-field approach for analyzing the dynamics in generic uncorrelated complex networks having arbitrary degree distributions. In scale-free networks with  $P(k) \sim k^{-\gamma}$ , a characteristic power-law exponent  $\gamma_c$  at which a dynamic transition takes place was identified. The voter model, which is similar to zero-temperature Glauber dynamics, has also been studied in complex net-

works [18–21]. Remarkably, some of these studies have indicated that the majority population in the initial state may not dominate asymptotically [21].

In these previous studies, the initial spins of the network vertices are considered to be randomly distributed, and it remains to be discussed whether a degree of order in the initial state affects the network dynamics. The measure of centrality has been used in network science to represent the characteristics of vertices relative to other vertices [9,22]. The use of centrality to characterize vertices thus allows a certain degree of order to be introduced into the initial spin configuration to which Glauber dynamics is applied. In the present study, the dependence of the dynamics on the initial spin configuration is examined using three kinds of centrality measures: degree centrality  $C_{deg}(v)$ , closeness centrality  $C_{close}(v)$ , and betweenness centrality  $C_{bet}(v)$  [23]. The clustering coefficient  $C_{clust}(v)$  is also considered as a characteristic of vertices [14]. These measures are defined for a vertex  $v$  as follows:

$$C_{deg}(v) = \frac{k_v}{N-1}, \quad (4)$$

$$C_{close}(v) = \frac{1}{\sum_t d_G(v,t)}, \quad (5)$$

$$C_{bet}(v) = \sum_{s \neq t \neq v} \frac{\sigma_{st}(v)}{\sigma_{st}}, \quad (6)$$

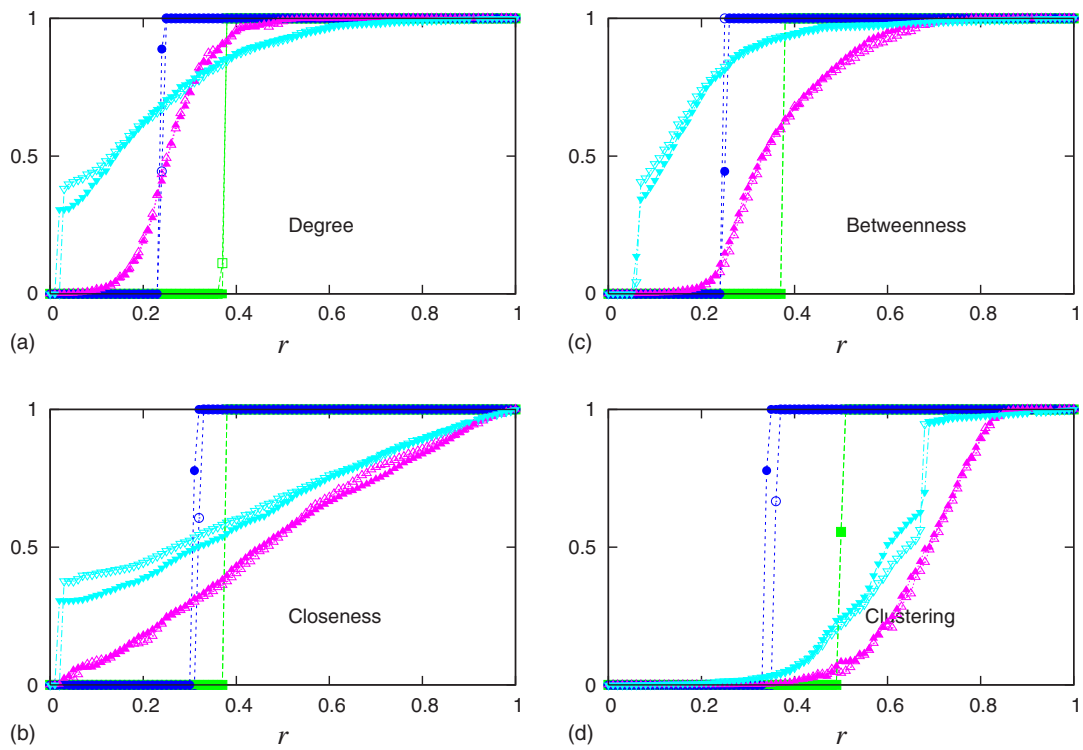


FIG. 2. (Color online) Convergence fraction of positive spins  $r'$  as a function of initial fraction  $r$ , derived from the initial ordering by (a) degree centrality, (b) closeness centrality, (c) betweenness centrality, and (d) clustering coefficient, on the random network ( $\square$ ), the BA network ( $\circ$ ), the KE network ( $\triangle$ ), and the CNN network ( $\nabla$ ). Open symbols and solid symbols denote data for system size  $N=18000$  and  $N=36000$ , respectively.  $\langle k \rangle=10$ .

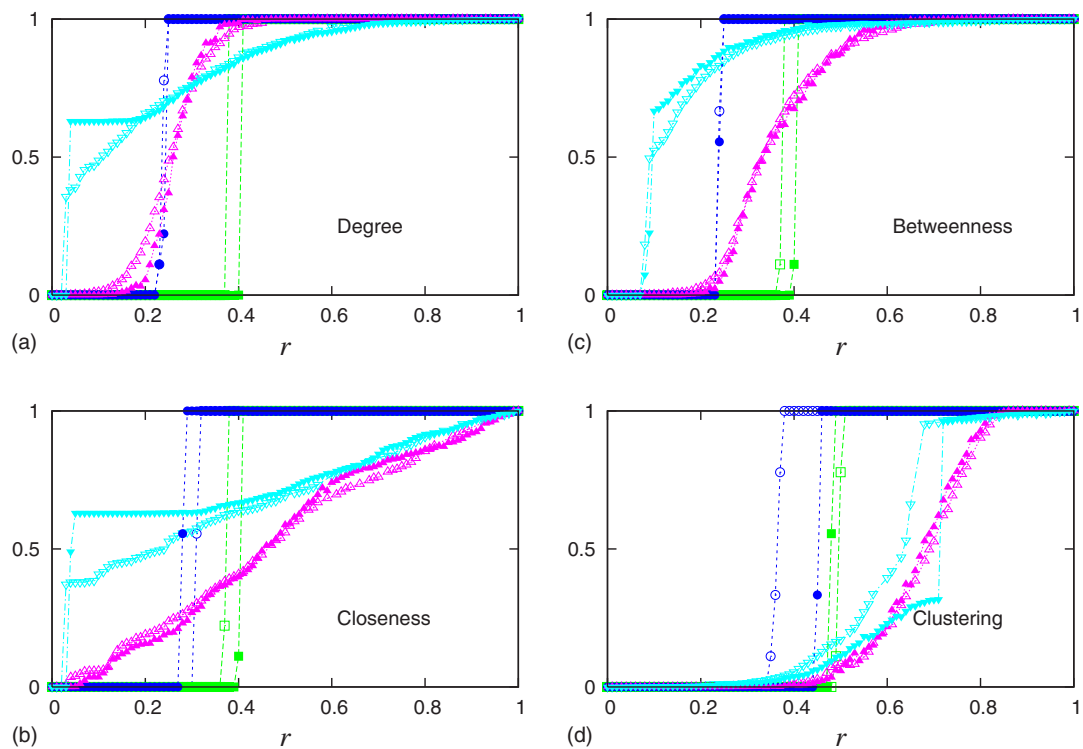


FIG. 3. (Color online) Same quantities and symbols as Fig. 2. Open and solid symbols denote the plot for the average degree  $\langle k \rangle=10$  and  $\langle k \rangle=18$ , respectively. The system size is  $N=9000$ .

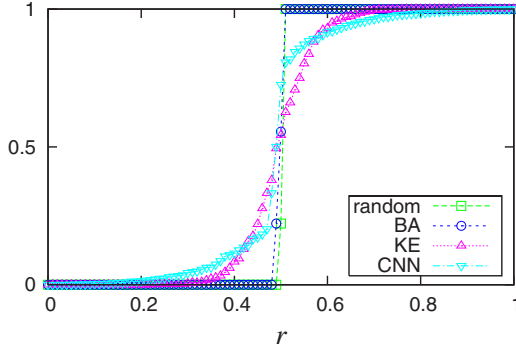


FIG. 4. (Color online) Convergence fraction of positive spins  $r'$  as a function of initial fraction  $r$  derived from a completely disordered (i.e., randomly distributed) spin state.  $N=18000$ ,  $\langle k \rangle=10$ .

$$C_{\text{clust}}(v) = \frac{E}{\binom{k_v}{2}}, \quad (7)$$

where  $k_v$  is the degree of the vertex  $v$ ,  $d_G(v, t)$  is the distance between vertices  $v$  and  $t$ ,  $\sigma_{st}$  and  $\sigma_{st}(v)$  are the number of shortest paths from  $s$  to  $t$  and from  $s$  to  $t$  via  $v$ , and  $E$  is the number of edges between the neighbors of vertex  $v$ .

Corresponding to the centralities of each vertex, the  $rN$  vertices with the largest centrality are assigned the positive spin state in the initial condition ( $\sigma(0)=+1$ ), while the remaining  $(1-r)N$  vertices are assigned  $\sigma(0)=-1$ . Here,  $r$  is the initial fraction of positive spins and  $N$  is the number of vertices in the network. Four types of complex network models are investigated using these initial conditions: The random graph network of Erdős and Rényi [15] and three types of structured scale-free networks—the Barabasi-Albert (BA) network [16], the Klemm-Eguíluz (KE) network (also referred to as the highly clustered scale-free network) [24], and the connecting nearest neighbor (CNN) network [25]. In the highly clustered scale-free network, the mixing parameter is set to  $\mu=0.1$ . All scale-free network models are categorized as evolving network models in which the distributions of the degree  $k$  follow a power-law distribution  $P(k) \sim k^{-\gamma}$ . However, the networks differ in that the BA network has a low clustering coefficient  $C \sim N^{-0.75}$  while the other two networks are much more highly clustered [ $C \sim O(1)$ ] and the CNN network has a positive (assortative) degree correlation. See Refs. [16,24,25] for further details. The same network models were used with a fixed number of vertices and average degrees in the preceding study [10]. In the present paper, the scaling effects of system size are also considered. In the series of simulations described below, all the vertices in the networks are confirmed to be *connected* by edges so that they form a single component.

The progression of positive spin fraction  $R_+(t)$  at  $t=\tau/N$ , starting from fixed initial fractions of  $r=0.4$  and  $r=0.7$  and a variety of initial orderings of positive spins, is shown in Fig. 1. The results shown throughout this report are averages obtained over several simulations. In the random and BA networks, the dynamics reaches a fully ordered state in which all of the vertices share the same spin state. With  $r=0.4$  in

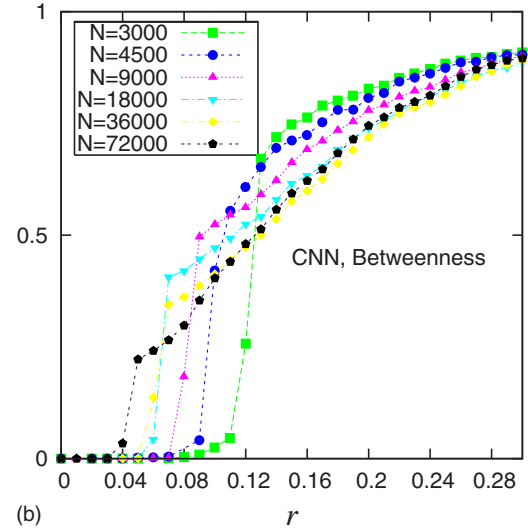
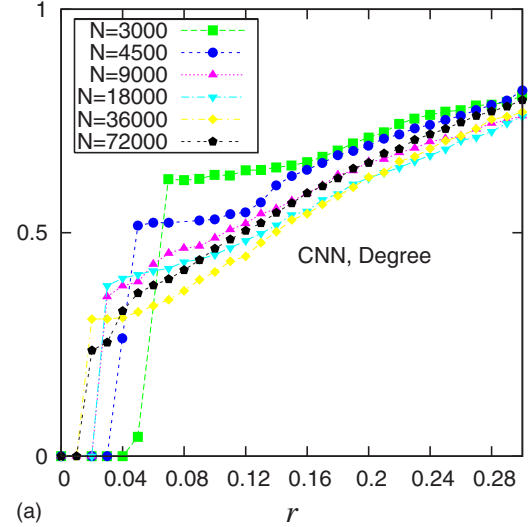


FIG. 5. (Color online) Convergence fraction of positive spins  $r'$  around the critical point  $r_c$  as a function of initial fraction  $r$  for the CNN networks of various sizes with initial orderings defined by (a) degree centrality and (b) betweenness centrality.

the initial state, positive spins derived from vertices having larger centralities, despite being the minority, spread over the entire network in all cases except the BA network with the initial ordering based on the clustering coefficient measure. In the KE and CNN networks, the dynamics reaches a metastable state consisting of two coexisting spin states with fraction  $r'$ , which differs from  $r$ .

Figures 2 and 3 show the relationships between the initial fraction  $r=R_+(0)$  and the convergence fraction  $r'=R_+(\infty)$  of positive spins in systems having various sizes and average degrees. Several classes of characteristic functional forms can be observed. If the respective numbers of two spin states are not equal (i.e.,  $r \neq 0.5$ ) and the two states are randomly distributed, it is obvious that the state having the larger quantity in the initial state will become dominant at  $t=\infty$ . This was verified by performing preliminary simulations (Fig. 4). In the random and BA networks, the convergence fraction of the two spin states exhibits a steplike change with respect to

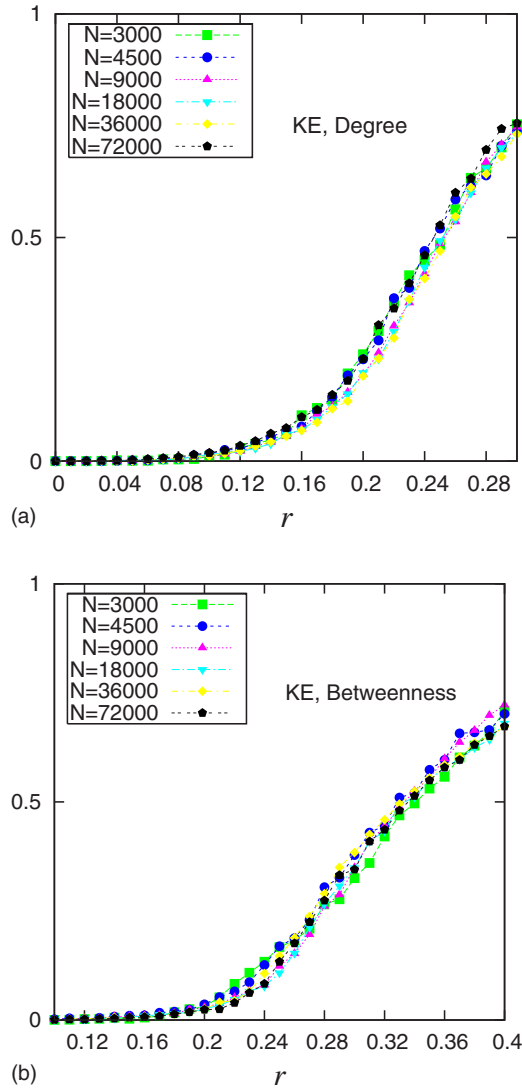


FIG. 6. (Color online) Same quantities and symbols as Fig. 5 for the KE networks.

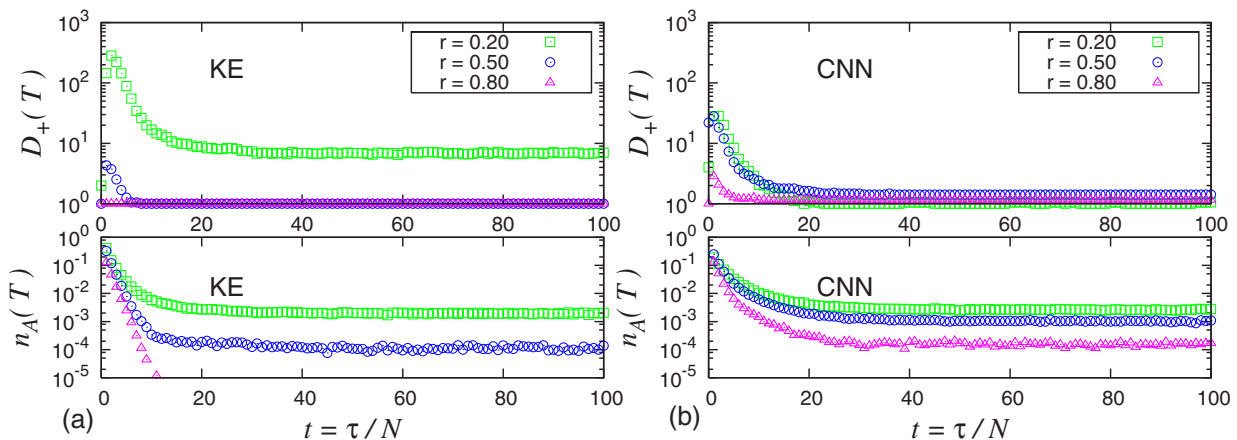


FIG. 7. (Color online) Evolution of the fraction of updating vertices (i.e., vertices that change state) (lower) and number of connected domains of positive spins (upper) on (a) KE and (b) CNN networks for  $r=0.20, 0.50,$  and  $0.80$ . Initial spins ordered by degree centrality.  $N=18000$ ,  $\langle k \rangle=10$ .

initial conditions, where the critical fraction  $r=r_c < 0.5$  is nontrivial. Notably,  $r_c$  is always lower in the BA network than in the random network.

In contrast, the KE and CNN networks produce much more complex variation in the convergence fraction of the two spin states with respect to ordering of the initial spin configuration, reflecting the emergence of a metastable state. The initial states with ordering defined by degree and betweenness centrality exhibit convex or sigmoidal functions over  $r$ , whereas the relationship for initial ordering defined by closeness centrality is almost linear. A critical point of  $r=r_c$  can also be identified in these traces, around which  $r'$  jumps or rapidly increases in some cases. If the initial spins are completely disordered,  $r_c$  should take a value of 0.5. However, the value of  $r_c$  differs according to the initial condition applied.

Figures 2 and 3 show the  $r-r'$  relationships for systems of various sizes and average degrees. Although some differences can be seen in the detailed shape of the relationship and the value of  $r_c$ , comparison of Figs. 2 and 3 suggests that the class of the characteristic function is independent of the system size and the average degrees, except in the case of the CNN network. To investigate the scaling effects in more detail, the critical behavior of the  $r-r'$  relationships was examined over a wider range of system size considering a representative set of initial orderings. A finite scaling on the CNN network can be recognized in both cases shown in Fig. 5. As the system size increases, the critical point  $r_c$  asymptotically approaches a certain value and the transition at  $r_c$  is not abrupt. No such scaling effects can be recognized on the KE network from Fig. 6, where the shape of the function and the value of  $r_c$  remains relatively constant, independent of system size.

The characteristics of the metastable state emerging in the KE and CNN networks are illustrated using the fraction of vertices that flip spin (i.e., the vertices that connect to an equal number of different spins or the vertices located at the boundary of a domain of identical spins) and the number of connected domains of positive spins at  $t=\tau/N$ . These parameters are denoted by  $n_A(t)$  and  $D_+(t)$ , respectively. Figure 7 shows an example of the variation in  $n_A(t)$  and  $D_+(t)$  with  $r$

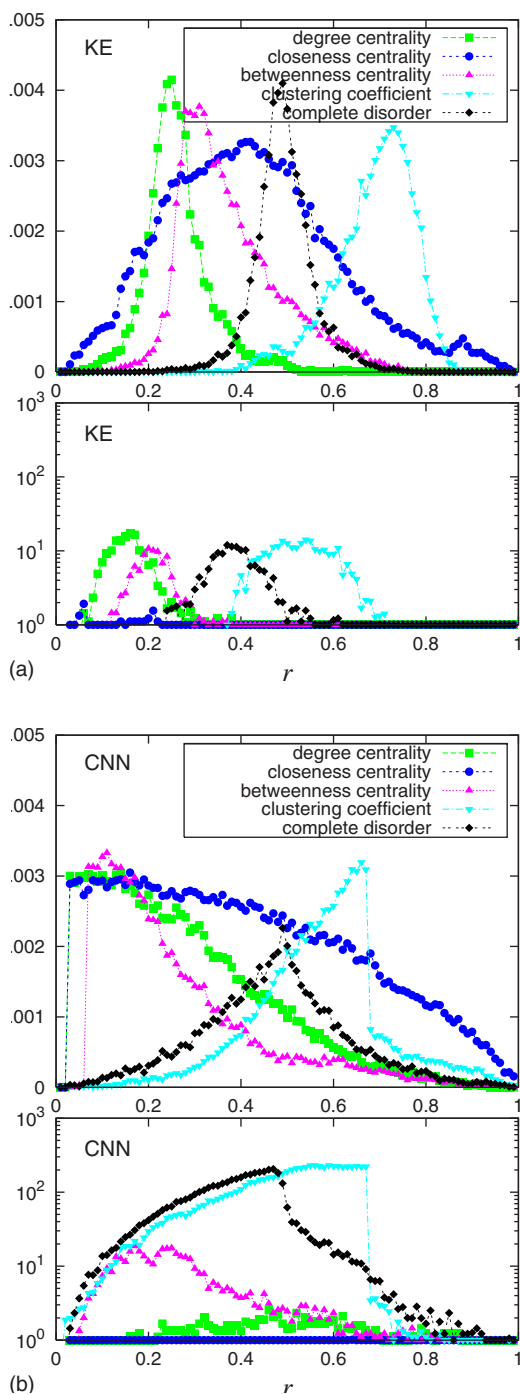


FIG. 8. (Color online) Fraction of updating vertices and number of connected domains of positive spin in the metastable state (at  $t = \infty$ ) with respect to  $r$  on (a) KE and (b) CNN networks considering various orderings of initial spins.  $N=18000$ ,  $\langle k \rangle=10$ .

in systems with initial ordering defined by degree centrality. It can be seen that  $D_+(t)$  does not always reach unity, suggesting that the metastable state can consist of several separated clusters of identical spins. Moreover,  $n_A(t)$  does not converge to zero, indicating that the metastable state is not static but rather a stationary active state. The results are essentially the same for other initial orderings. It has been reported that a similar phenomenon might occur in a random

network from a completely disordered initial state [7], implying that the final state consists of two large domains of opposite spin. A similar investigation was conducted using high-dimensional lattices [13]. The present simulations considering a range of complex network structures and arbitrary initial conditions suggest that an alternative outcome is possible.

The plots of  $n_A(t)$  and  $D_+(t)$  in the final metastable state (at  $t = \infty$ ) with respect to  $r$  for various initial conditions (Fig. 8) indicate that the final states for the two networks are quite different. On the KE network,  $n_A(\infty)$  exhibits a clear peak for each set of initial conditions, whereas a more complicated noncontinuous function appears on the CNN network. For a completely disordered initial configuration,  $n_A(\infty)$  reaches a maximum at  $r=0.5$  and forms a symmetric function. For most initial conditions on both networks, the initial fraction  $r$  affording the maximum  $n_A(\infty)$  is consistent with the critical fraction  $r=r_c$  in Fig. 2. It is therefore safe to conclude that the metastable state arising from  $r \approx r_c$  is somewhat unstable. The number of connected domains of positive spins (Fig. 8) reveals another characteristic. At small  $r$ , positive spins tend to shrink and become extinct, and thus  $D_+(\infty)=0$ . As  $r$  increases, several clusters of positive spin are able to persist, causing  $D_+(\infty)$  to increase. As  $r$  increases further, the fragmented domains begin to agglomerate to form fewer, larger clusters, until a single large domain emerges [i.e.,  $D_+(\infty)=1$ ]. The degree of fragmentation depends on the initial conditions and the networks. It should be noted that  $D_+(\infty)$  is always equal to or very close to unity in both networks when the order of the initial spin configuration is defined by the closeness centrality measure. It is suspected that this gives rise to the linear relationship between  $r$  and  $r'$ , as shown in Fig. 2(b).

In summary, the effect of the initial spin configuration on zero-temperature Glauber dynamics acting on various types of complex networks was examined. Through a series of numerical analyses it was revealed that nontrivial diffusive behavior that depends on both the initial condition and the network structure occurs in such systems. In some cases, the final dynamics reached a metastable state involving two co-existent spin states with several connected domains of identical spins, the marginal vertices of which flip continuously. The functional relationship between the initial fraction  $r$  of positive spins and the final fraction  $r'$  can be categorized into one of several patterns, some of which exhibit critical point  $r_c$ . The patterns and criticality were found on networks of various system sizes and average degrees. A finite scaling effect with respect to system size was identified on the CNN network, whereas the other networks examined exhibited no such scaling effect.

Although a clear understanding of the origin of the behavior revealed in this study has yet to be obtained, some observations can be made on the general characteristics of these phenomena. As the KE and CNN networks are known to be more highly clustered than the random and BA networks, the final metastable state may originate from the topological cohesiveness of such networks. However, the characteristic functional relationships between  $r$  and  $r'$ , and the origins of

the critical point  $r_c$ , remain unclear. One possible feature that could be responsible for these characteristics is the correlation in networks [26]. The CNN network is known to have a positive degree correlation [25], and hence if the vertices of higher degree are more likely to be interconnected, the positive spins originating from these vertices will not disappear, even for a small initial fraction. For other initial orderings, the correlation of centralities may also play a key role in the unique dynamics observed. The results of the present study may therefore lead to novel understandings of the underlying

characteristics of individual vertices measured thus far by centralities. It is also expected that further characteristics of network topologies remain to be revealed. The nonequilibrium spin dynamics exhibited by these systems does not appear to be comprehensively explainable based on topological scaling or conventional measures of the underlying complex networks alone. Nevertheless, the present study has revealed a strong dependence of spin dynamics on the initial conditions determined by the topological heterogeneity of the network.

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